INTERDICTION CONSIDERATIONS IN LEONTIEFF-TYPE MODELS OF LAND LOGISTIC NETWORKS

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Letter on cover:

INTERDICTION CONSIDERATIONS IN
DECISION-TYPE MODELS OF LAND
LOGISTIC NETWORKS

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ABSTRACT

This paper consists of an analysis of the problems pertaining to the interdiction of a land logistic network. Leontieff-type input-output models are employed to describe the flows of materiel in the logistic system. The objective of interdiction is regarded to be the reduction of exogenous demands required of the system. These demands are represented in the models. Consideration is given to the alternatives of interdicting the flows of materiel through the depots, the materiel in the stockpiles at the depots, the flows of materiel between depots, and of interdicting the handling facilities and capacities of the depots, the stockpiles, and the logistic routes. The paper contains a discussion of the use of analog computers and associated plotting equipment for the solution of the systems of differential equations involved in the models. The author points out the utility of these computers to the military commander using the models in planning an interdiction campaign.

PREFACE

Advances in airpower have increased the scope and importance of the interdiction of enemy logistic routes, depots, and stockpiles in the military commander's concept of operations, and conceivably, the role of interdiction may become commensurately more important with advances in guided missiles. It is in regard to these prospects that this paper is concerned with the development of a technique which will be useful in planning interdictory campaigns.

The primary problem considered in this paper is the planning of an interdiction campaign against an enemy logistic system within the enemy zone of interior and its contiguous area of operations. The planning of the interdictory campaign is contingent upon the mathematical models which describe the flows of munitions and materiel in the complex of logistic routes, depots, and stockpiles of the zone of interior and the front. The models are essentially the same as those developed by Wassily W. Leontieff to study national economics. However, the models presented in this paper are utilized to describe logistic flows and do not involve the economic considerations of an interdictory campaign.

The accuracy of these models in estimating the enemy's logistical situation is dependent on the intelligence that can be accumulated concerning the enemy logistic system and

its operation. It may be anticipated that the lack of sufficient intelligence will severely limit the usefulness of these models, as well as other models, as a planning tool. The use of large-scale analog computers has been found to be of considerable value in manipulating the mathematical quantities of the models, and the use of these computers is strongly recommended.

The author emphasizes that the models developed in the text are intended to be planning implements and do not necessarily constitute a quantitative analysis of the situation as the main attack progresses.

This thesis was written during the period January-June, 1956, at the U.S. Naval Postgraduate School, Monterey, California. I wish to express my appreciation to Professor Thomas E. Oberbeck for his technical assistance and wholehearted interest in the thesis, to Professor J. Giarrantana, who offered many helpful suggestions as second reader, and to Mrs. Mary E. New for careful clerical assistance.

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TABLE OF SYMBOLS

(Listed as the symbol appears in the text)

t_o	The time at which the enemy logistic system is taken under study.
t_1	The time at which the attacking commander launches an attack in the area of operations.
m	The number of depots in the enemy's logistic system.
i	A particular depot of the m depots in the system. $i = 1, 2, ---, m$.
$X_i(t)$	The flow of materiel through the i^{th} depot at any time, t, measured in cargo-tons per month.
$\dot{S}_i$	The flow of materiel to or from the stockpile at the i^{th} depot at any time, t, measured in cargo-tons per month.
$x_{ki}(k \neq i)$	The flow of materiel that is sent from the i^{th} depot to the k^{th} depot at any time, t, measured in cargo-tons per month.
a_{ki}	A proportionality constant associated with a logistic route.
$\dot{X}_i(t)$	The time rate of change of the flow of materiel through the i^{th} depot at any time, t, measured in cargo-tons per month squared.
b_i	A proportionality constant associated with a stockpile flow measured in units of time, t.
$S_i(t)$	The size of the stockpile at the i^{th} depot at any time, t, measured in cargo-tons of materiel.
S_i^o	The size of the stockpile at the i^{th} depot at time t_o, measured in cargo-tons of materiel.
X^o_i	The flow of materiel through the i^{th} depot at time t_o, measured in cargo-tons per month.
$\hat{X}_i$	The maximum flow of materiel through the i^{th} depot that is possible. This quantity is limited in magnitude by the capacity of the handling facilities at that depot, and is measured in cargo-tons per month.

$\hat{S}_i$ — The maximum size of the stockpile at the i^{th} depot, measured in cargo-tons, that can be accumulated because of stowage facilities.

$\hat{x}_{ki} (k \neq i)$ — The maximum flow of materiel that can be sent from the i^{th} depot to the k^{th} depot, measured in cargo-tons per month.
This quantity is limited by the handling capacity of the one-way logistic route from the i^{th} depot to the k^{th} depot.

$[X]$ — A column matrix: $\begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_m(t) \end{bmatrix} = [X]$

$[C]$ — A composite matrix of constants $[C] = -[B]^{-1}[M]$ where:

$$[B] = \begin{bmatrix} b_1 & 0 & - & - & - & 0 \\ 0 & b_2 & - & - & - & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & - & - & - & b_m \end{bmatrix}$$

$$[M] = \begin{bmatrix} 1 & 0 & 0 & - & - & 0 \\ 0 & 1 & 0 & - & - & 0 \\ 0 & 0 & 1 & - & - & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & - & - & 1 \end{bmatrix} - \begin{bmatrix} 0 & a_{21} & a_{31} & - & - & -a_{m1} \\ a_{12} & 0 & a_{32} & - & - & -a_{m2} \\ a_{13} & a_{23} & 0 & - & - & -a_{m3} \\ \vdots & \vdots & \vdots & & & \vdots \\ a_{1m} & a_{2m} & a_{3m} & - & - & 0 \end{bmatrix}$$

$[X^o]$ — A column matrix: $\begin{bmatrix} X_1^o \\ X_2^o \\ \vdots \\ X_m^o \end{bmatrix} = [X^o]$

x_{ni} — The demand function for the i^{th} depot. The flow of materiel to the forces in the field, measured in cargo-tons per month, at the i^{th} depot.

α_{ni}	The value of the demand function at time t_1.
m_{ni}	The slope of the linear portion of the curve shown in Figure 3(b).
β_{ni}	The exponential exponent of the curve shown in Figure 3(c).
$[x]$	A column matrix of demand functions: $\begin{bmatrix} x_{n1} \\ x_{n2} \\ \vdots \\ x_{nm} \end{bmatrix} = [x]$
x^*_{ni}	The reduced value of the demand function at the i^{th} depot, which will result from interdiction.
P_{ni}	A fractional constant associated with the reduced value of the demand function. $0 \leqq P_{ni} < 1$.
R_{ni}	The amount by which the demand function at the i^{th} depot is reduced, i.e., $x^*_{ni} = x_{ni} - R_{ni}$.
$R[X_i(t)]$	The amount by which the flow through i^{th} depot is reduced with time.
$X^*_i(t)$	The reduced flow of materiel through the i^{th} depot, at any time t measured in cargo-tons per month.
$x^*_{ki}(k \neq i)$	The reduced flow of materiel that is sent from the i^{th} depot to the k^{th} depot, at any time t measured in cargo-tons per month.
$\dot{S}^*_i(t)$	The reduced flow of materiel that is sent to or from the stockpile at the i^{th} depot at any time t measured in cargo-tons per month.
$\overline{X}_i(t)$	The reduced maximum flow of materiel through the i^{th} depot at any time t, measured in cargo-tons per month. This reduced maximum flow is caused by interdicting the handling facilities at the i^{th} depot. $\overline{X}_i$ is defined by: $\overline{X}_i = \hat{X}_i - \phi_i$, where ϕ_i is the amount by which the maximum flow is reduced due to interdiction.
$\overline{x}_{ki}(k \neq i)$	The reduced maximum flow of materiel that is sent from the i^{th} depot to the k^{th} depot at any time t measured in cargo-tons per month. This reduced maximum flow is caused by interdicting the one-way logistic route facilities between the i^{th}

depot and k^{th} depot, and $\bar{x}_{ki}$ ($k \neq i$) is defined by: $\bar{x}_{ki} = \hat{x}_{ki} - \psi_{ki}$, where ψ_{ki} is the amount by which the maximum flow is reduced due to interdiction.

$\bar{S}_i(t)$ — The reduced maximum size of the stockpile at the i^{th} depot at any time t, measured in cargo-tons. This reduced maximum size of the i^{th} stockpile is caused by interdicting the facilities and storage capacity of that stockpile. $\bar{S}_i(t)$ is defined by $\bar{S}_i(t) = \hat{S}_i - \theta_i$, where θ_i is the amount by which the maximum size of the stockpile is reduced over time due to interdiction.

$f(\lambda)$ — The characteristic matrix of C, namely, $f(\lambda) = \lambda I - C$.

$F(\lambda)$ — The adjoint matrix of $f(\lambda)$.

$\Delta(\lambda)$ — The characteristic function of the matrix C, namely, $\Delta(\lambda) = |f(\lambda)|$.

λ_r — The characteristic roots $\Delta(\lambda)$.

CHAPTER I

A NON-TECHNICAL DISCUSSION

1. Introduction.

The term "interdict" is defined in OPNAVINST 3020-1A [1] [1] as:

> "To prevent or hinder, by any means, enemy use of an area or route."

To interdict enemy logistic routes would imply that the enemy would be hindered in the use of his lines of supply thereby dissociating his operating military forces from their bases of operations. Since the enemy's munitions and reinforcements flow along these logistic routes, that flow would be restricted commensurately with the amount of damage inflicted on his logistic routes and carriers. If the interdiction is continued over a period of time, the enemy's operating forces may experience shortages of supplies, munitions and reinforcements. This may be regarded as part of the objective of an interdictory campaign against the enemy logistic system.

Interdictory campaigns are by no means new in the military commander's concept of operations. However, the weapons methods utilized to attain the objective in interdictory campaigns have been advanced considerably since the advent of the long range aircraft, the submarine, and other such weapon systems as interdictory weapons.

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1. Numbers in brackets refer to references cited in the Bibliography

Historically, the siege of fortresses in ancient times often entailed no more than an interdictory campaign. In such a campaign the objective would be to starve the defending forces into submission by surrounding the fortress and thereby preventing the entry of supplies and munitions to the fortress. Eventually, stockpiles within the fortress would dwindle and the beleaguered defenders would either have to submit to the opposing forces or engage them in battle. Such campaigns often did not involve a direct assault on the fortress, and since time was not of great importance, the campaigns would last for long periods of time.

In wars where the enemy depended primarily on sea routes for his logistic routes, naval blockade proved to be an important interdictory weapon. Examples of such interdictory campaigns are the naval blockade of the Confederate States by the Union forces during the Civil War, the German U-boat campaigns of World Wars I and II, and the United States submarine and mining campaigns against the Japanese in World War II. In the Korean Action the United Nations Command interdicted the enemy's logistic routes and the transporting vehicles by the use of aircraft and naval vessels. Although the aircraft interdictory campaign conducted in Korea was somewhat more limited in scope than would normally be expected, in general, even extensive aircraft interdictory campaigns do not constitute strategic air warfare. These campaigns usually are

component parts of strategic air warfare, but they do not include all of the broad concepts involved in such warfare.

2. Concept of a Logistic System.

It is appropriate at this stage to define a land logistic system in rather broad terms. A land logistic system is envisioned as being contained in a large geographical area constituting a zone of interior and its associated area of operations. The system will consist of depots, either cities or handling junctions, which are interconnected by a complex of logistic routes over which flow the supplies and munitions of war. Also associated with the system are the components such as stockpiles and the carriers of cargo; the trucks, vessels, trains, and aircraft. Obviously, this definition of a land logistic system is limited, but it will suffice as a qualitative description for the purposes of this paper.

3. Scope of the Thesis.

This paper presents an elementary treatment of the interdiction problem in regard to an interdiction campaign against a land logistic complex. While it is not the most sophisticated treatment, it does provide the military planner with a practical basis upon which he may build a more complicated and comprehensive interdictory plan. The interdictory aspects dealt with in this paper are primarily:

a. The adaptation of a simple mathematical model to describe the aggregated supply and munitions flows in an enemy complex of logistic depots, routes, and stockpiles.

b. The modification of this model to describe the logistic flows in the enemy logistic complex when the associated front is subjected to attack.

c. The formulation and brief discussion of the problem confronting the attacking commander when he interdicts the enemy's logistic system in an attempt to cause the system to fail to supply the required quantity of supplies and munitions at the frontal depots.

4. The Planning Implements.

A rather broad qualitative description of a land logistic system has been proposed in Section 2. To describe such a system in detail would be a monumental task, and although it would be possible to do so, an attempt to describe the system quantitatively would yield a model of such complexity that it would be virtually unusable by the attacking commander. For this reason, the quantitative mathematical models presented in this thesis deal only with those aspects of a land logistic system which concerns the interdictory commander, i.e., the handling facilities for the flows of supplies and munitions in the system and the flows themselves in terms of the flows of materiel through the depots, along the logistic routes, and into and out of the logistic stockpiles. These mathematical models are proposed as the planning implements which will assist the interdictory commander in allocating his interdictory forces. Chapters II, III and IV present the rudiments of this planning implement.

In particular, Chapter II deals with a model which describes the flows of supplies and munitions in a land logistic system when there is no attack on the system in the area of operations. The model is modified in Chapter III to describe the flows in the same system when the system is subjected to an attack in the area of operations. Chapter IV formulates and discusses various facets of the interdiction problem based on the modified model of Chapter III.

5. Summary.

The mathematical models developed in Chapters II and III have certain limitations which limit the usefulness of these models in describing the actual materiel flows in a logistic system. At best, the models can describe only approximately the actual flows of materiel in the system. Also, the lack of intelligence concerning the enemy's logistic system severely limits the usefulness of these or any other models. In many cases, it may be necessary to estimate the magnitudes of certain materiel flows when accurate data on these flows are lacking. Although the author assumes throughout this paper that complete information concerning the enemy's logistic system is available to the attacking commander, it should be realized that such cases would be uncommon. In any case, the purpose of this paper is concerned rather with the concept of the model than its accuracy in an actual application.

The discussion in Chapter IV indicates the immense magnitude of the interdiction problem which would confront the military planner in planning an interdiction campaign. The interdictory aspects discussed in that chapter are based on the use of the modified model of Chapter III to describe the materiel flows in the enemy's logistic system. Chapter V embodies a brief but important discussion on the use of analog computers to solve some of the systems of equations developed in the mathematical models of Chapters II and III.

CHAPTER II

THE BASIC MODEL

1. Explanation of the Model.

A Leontieff-type model [2] is introduced in this chapter to represent the flow schematic of a land logistic system. From the time that a study of the system is undertaken, time t_o, until a later time, t_1, it is assumed that there are no supplies or munitions being consumed in defending the area in which the system is located. The logistic systems of the zone of interior and of the area of operations will be considered to be parts of the same system and for this reason, they are not distinguishable in the model. The reader is again reminded of the definition of a land logistic system in the sense that there are numerous depots of the system which are interconnected by a complex of logistic routes and that there are logistic stockpiles associated with the depots. The model describes a relationship between the flows of supplies and munitions in this system.

A schematic diagram of the logistic flows is shown in Figure 1 for a two depot system. In general, there will be some larger number of depots, say m all of which are interconnected by flows in a manner similar to that of the two-depot case. In the general case, let us consider the i^{th} depot. At any particular time, there will be a certain flow of materiel through the depot, and this flow is denoted

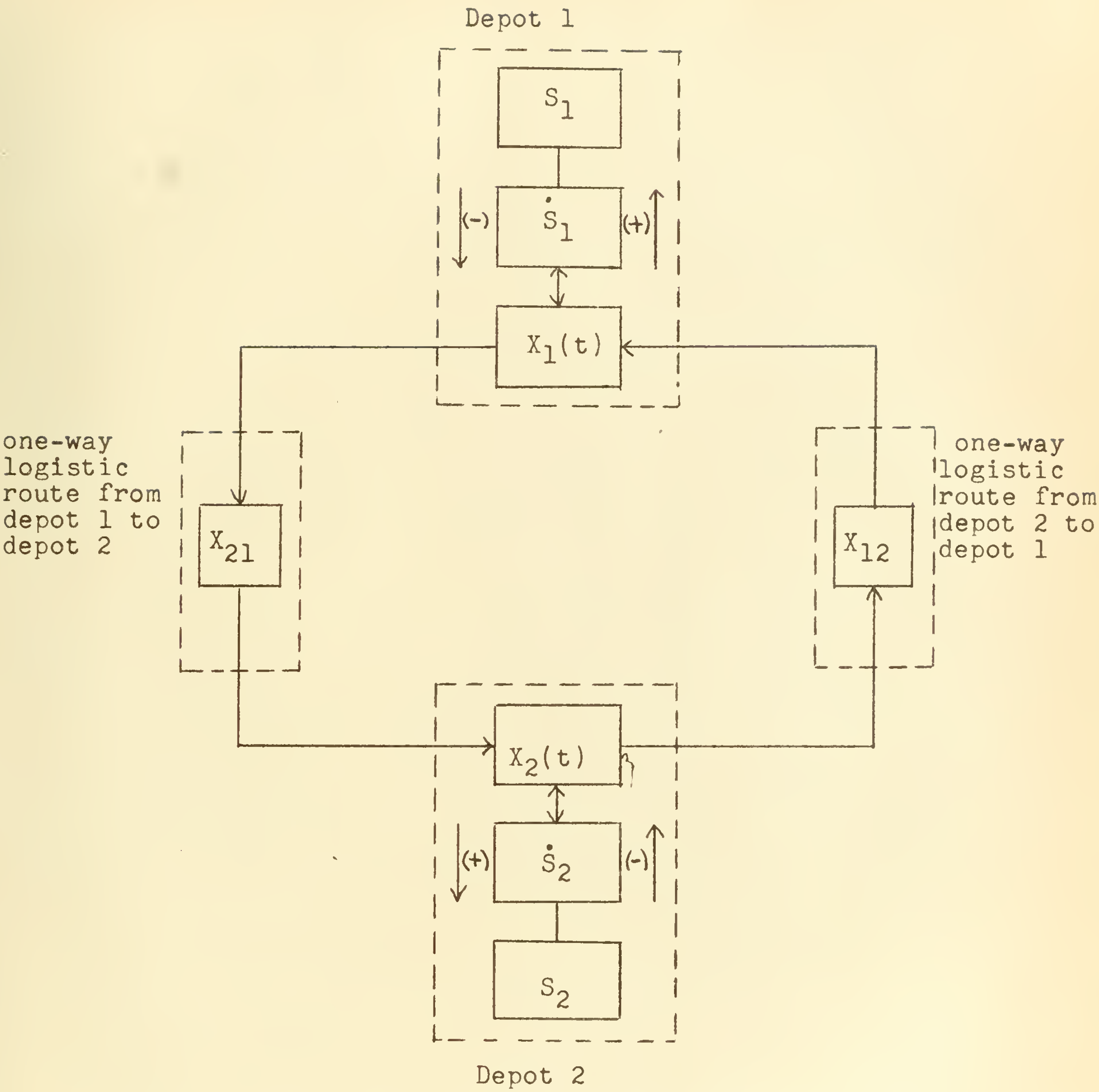

Figure 1. Schematic Diagram of Logistic Flows (Two-depot Sketch)

as $X_i(t)$. That part of this flow that is sent to or from the depot stockpile is denoted by $\dot{S}_i$, and that part that is sent to other depots for further disposition, by

$$\sum_{\substack{k=1 \\ k \neq i}}^{m} x_{ki} .$$

More precise definitions of these quantities appear in the Table of Symbols. Each of the m depots will have a set of these flows.

2. The Equations of Flow.

The flows of materiel in the system are related to each other by a set of m equations, one equation for each depot. These equations are of the type:

$$X_i(t) - \sum_{\substack{k=1 \\ k \neq i}}^{m} x_{ki} - \dot{S}_i = 0, \; i = 1, 2, ---, m. \qquad (2\text{-}1)$$

However, these m equations are subject to certain structural relationships and restrictions. It is assumed that the portion of the flow through the i^{th} depot that is sent to the k^{th} depot, i.e., the quantity x_{ki}, is proportional to the flow through the k^{th} depot. This assumption is expressed mathematically by equations of the type:

$$x_{ki} = a_{ki} X_k(t), \; i \neq k, \qquad (2\text{-}2)$$

where the coefficients, a_{ki}, are constants, and

$$0 \leqq a_{ki} \leqq 1.$$

Another fundamental assumption of the model concerns the flow into or out of the i^{th} stockpile, the quantity $\dot{S}_i$. It is assumed that the quantity $\dot{S}_i$ is proportional to the

rate of change of the flow of materiel through the depot, the quantity $\dot{X}_i(t)$. This assumption gives rise to the additional structural equations of the type:

$$\dot{S}_i = b_i \dot{X}_i(t), \quad i = 1, 2, ---, m, \qquad (2\text{-}3)$$

and in which the coefficients, b_i are constants. Integrating equations (2-3) results in the equations:

$$S_i = \overset{o}{S}_i + b_i (X_i(t) - \overset{o}{X}_i), \quad i = 1, 2, ---, m \qquad (2\text{-}4)$$

where the $\overset{o}{S}_i$ and $\overset{o}{X}_i$ are the initial values of S_i and $X_i(t)$, i.e., the values of S_i and $X_i(t)$ at the time t_o. Equations (2-4) will give the size of the stockpiles at any time subsequent to time t_o. These stockpiles will be accumulated ($\dot{S}_i > 0$) or decumulated ($\dot{S}_i < 0$) over a period of time depending on the direction of the flows, $\dot{S}_i$.

By invoking the two fundamental assumptions just discussed, it is possible to restate equations (2-1) as:

$$X_i(t) - \sum_{\substack{k=1,\\ k\neq i}}^{m} a_{ki}X_k(t) - b_i\dot{X}_i(t) = 0, \quad i = 1,2,- -,m \qquad (2\text{-}5)$$

Thus far, the flow of materiel over the logistic routes, $x_{ki}(k \neq i)$, the size of the stockpiles, S_i, and the flow of materiel through the depots, $X_i(t)$, are described by the sets of equations, (2-2), (2-4) and (2-5), respectively. These are the variables which will concern us mostly.

3. Restrictions on the Variables.

The primary variables, $x_{ki}(k \neq i)$, S_i, and $X_i(t)$ mentioned in the previous section are all subject to certain constraints consistent with the physical limitations of the logistic system. Consider first the flow of materiel through the i^{th} depot, $X_i(t)$, which depends on the handling facilities of the depot. This flow can range from zero, in the case where the depot cannot handle any cargo, to an estimated maximum value, $\hat{X}_i$, the maximum limit of flow of materiel that the depot is physically capable of handling. At the time when the system is taken under study, it is assumed that the system is already functioning and that the initial flow of materiel through the depot at that time is X_i^o. It will suffice to say that regardless of the trend that it may subsequently take, $X_i(t)$ starts at a value X_i^o and is thereafter restricted to values between zero and $\hat{X}_i$. This condition is expressed by the inequality:

$$0 \leqq X_i(t) \leqq \hat{X}_i,$$

where,

$$X_i(t_o) \equiv X_i^o \leqq \hat{X}_i.$$

In a like manner the amount of materiel in a stockpile, represented by the variable S_i, being dependent on $X_i(t)$, starts at a value, S_i^o, and is restricted to values between zero and $\hat{S}_i$, where $\hat{S}_i$ is the maximum amount of materiel that can be stored at the i^{th} depot because of the storage capacity at that depot. Similarly as before:

$$0 \leqq S_i \leqq \hat{S}_i \tag{2-7}$$

where,

$$S_i(t_o) \equiv \overset{o}{S_i} \leqq \hat{S}_i$$

Furthermore, the one-way flow of materiel over the logistic routes, the quantity x_{ki} $(k \neq i)$, will range in value from zero to a value $\hat{x}_{ki}$ $(k \neq i)$, determined by the maximum one-way route capacity for that particular route, or:

$$0 \leqq x_{ki} \leqq \hat{x}_{ki} \tag{2-8}$$

where,

$$x_{ki}(t_o) \equiv \overset{o}{x}_{ki} \equiv a_{ki}\overset{o}{X_i} \leqq \hat{x}_{ki}, \; k \neq i.$$

These flows are somewhat analogous to the flow of water through a pipe. For example, $X_i(t)$ could be considered as the flow of water flowing through the i^{th} depot pipe at any particular time. The flow can range from no flow to a maximum flow, $\hat{X}_i$, determined by the dimensions of the pipe.

4. Operation of the Model.

A solution of the equations (2-5) in matrix form is derived in Appendix A. This solution is:

$$\left[X\right] = e^{-\left[c\right](t-t_o)} \left[X^o\right]$$

The variables $x_{ki}(k \neq i)$ and S_i can be found in terms of the elements, $X_i(t)$, of the matrix $\left[X\right]$, and thereby illustrate the relationships that exist between $x_{ki}(k \neq i)$, S_i, and $X_i(t)$. Due to the reciprocity of the variables in the model, whenever any one of these variables reaches a

limiting value, the effect is reflected throughout the model to the other variables. In operation, the model may function in such a manner that the variables remain within their specified limits at all times. We shall designate this situation as the unlimited case. As the other alternatives, the model may be limited in operation by one of the primary variables, $X_i(t)$, $x_{ki}(k \neq i)$, or S_i reaching one of its limits first on a time scale. These occurrences will be called the depot capacity bound case, the route bound case, and the stockpile bound case, respectively. These cases will be discussed in more detail in Chapter IV using the modified model of Chapter III.

CHAPTER III

THE MODIFIED MODEL

1. General Discussion.

A mathematical model was developed in Chapter II which describes the various logistic flows in an enemy logistic system for the case where there is no attack in the area of operations. In this chapter, the model will be modified by the introduction of additional flows of materiel in the system assumed to result from an attack on the system. When an attack is initiated by the opposing forces in the area of operations at time t_1, one or more of the depots that are regarded as being in the area of operations must establish additional materiel flows. These flows provide the materiel which is consumed in defending the environs of the depots concerned from the attack. Figure 2 shows the placement of these new flows, x_{ni}, in the schematic flow diagram of a frontal depot.

2. The Demand Functions.

The new flows, x_{ni}, of the frontal depots are the consequences of an attack on these depots by the opposing forces and will be assumed to be dependent on the magnitude of the initial assault and the rate of build-up of the attack. They will be called the demand functions. Let us examine some typical forms of these demand functions.

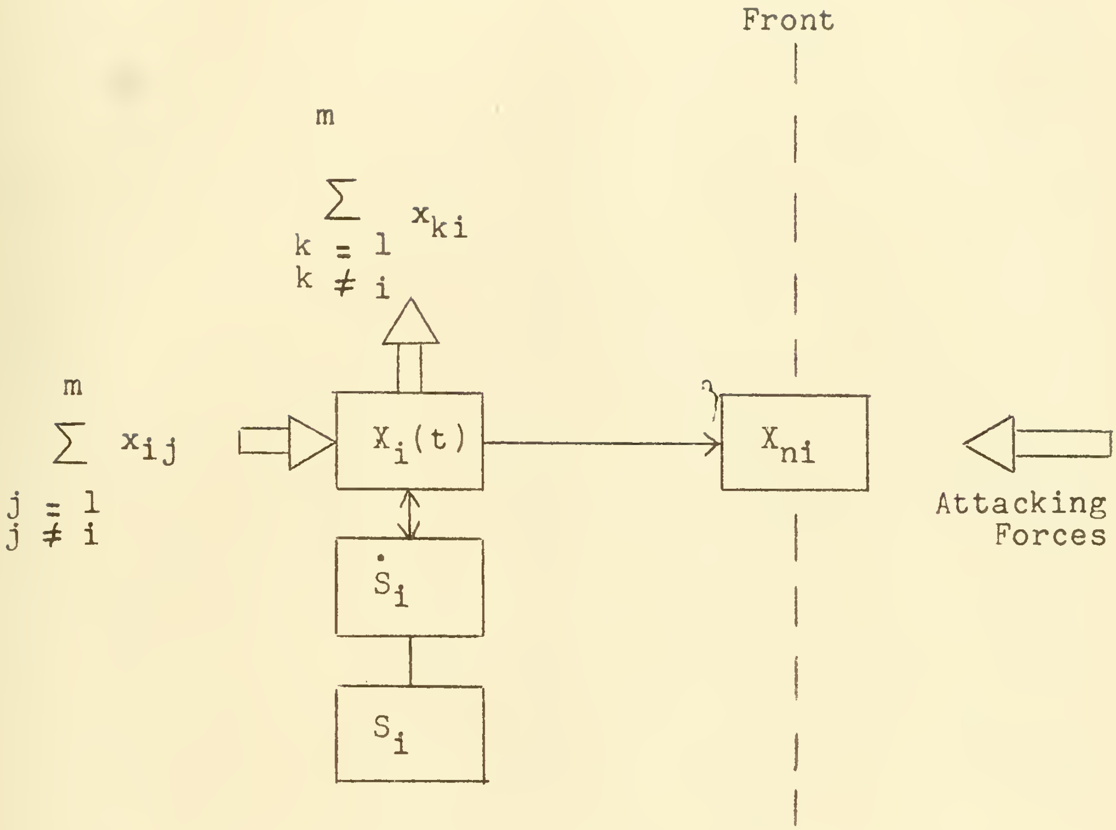

Figure 2. Schematic Flow Diagram of a Frontal Depot when Subjected to an Attack

Since the demand functions are dependent on the magnitude of the initial assault, the functions will acquire an initial value, α_{ni}, at time t_1, for each of the depots assaulted. The manner in which the functions vary after time t_1 depends on the rate of build-up of the attack. To describe all of the different variations in the build-up of the attack is beyond the scope of this paper. However, typical demand functions are shown graphically in Figure 3. Figure 3(a) is a graph of a demand function when the demand caused by the assault reaches immediately a value α_{ni}, and then remains constant throughout the remainder of the attack. Figure 3(b) shows the graph for the case where there is a linear build-up of demands beginning with the initial assault, and Figure 3(c) is for the case where there is an accelerating build-up. The mathematical forms of demand functions with these characteristics are:

$$x_{ni} = \alpha_{ni} \text{ (a constant)},$$

$$x_{ni} = \alpha_{ni} + m_{ni}(t-t_1), \tag{3-1}$$

and $$x_{ni} = \alpha_{ni}\, e^{\beta_{ni}(t-t_1)}, \text{ respectively.}$$

3. The Modified Model.

To modify the mathematical model of Chapter II, we need only to insert the demand functions and note that values of the variables of the previous model at time t_1 are not the initial values of those same variables under the new regime. For example, $X_i(t)$ will not necessarily follow the same

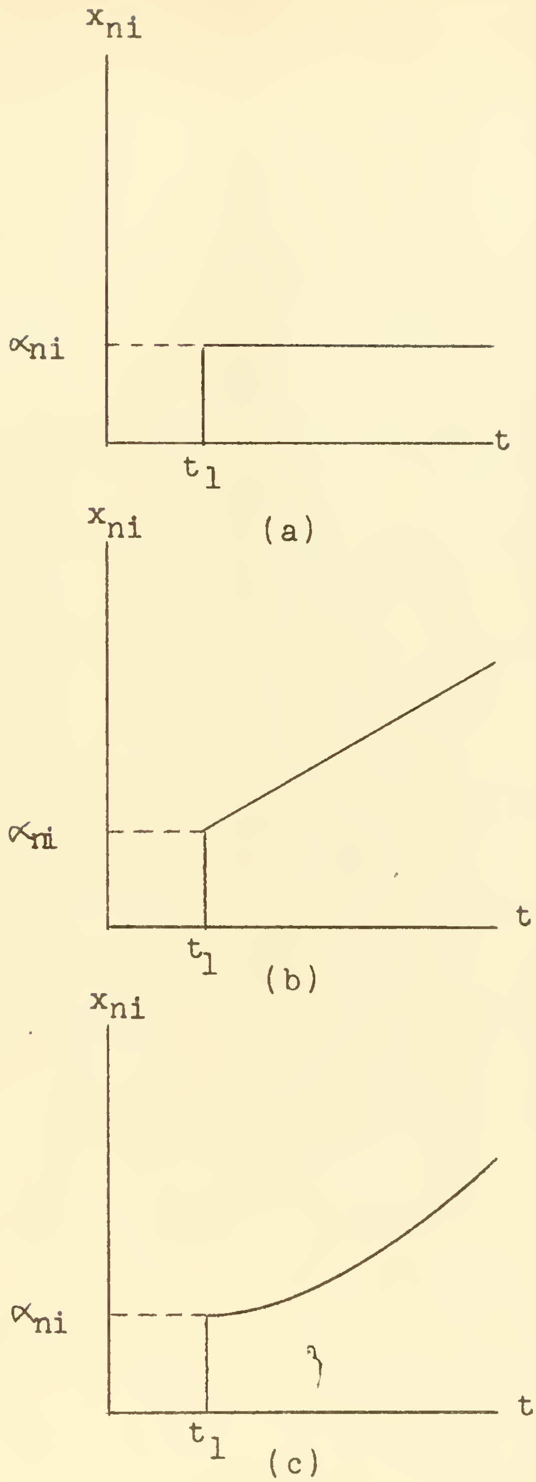

Figure 3. Typical Forms of Demand Functions.

trend as it had previously, but will start at some new value, $X_i(t_1)$, and may follow an entirely different trend after time t_1. The demand functions, x_{ni}, whatever their form may be, are introduced in the m equations (2-5) to give the modified model:

$$X_i(t) - \sum_{\substack{K=1 \\ K \neq i}}^{m} a_{ki} X_k(t) - b_i \dot{X}_i = x_{ni}, \quad i = 1, 2, ---, m. \tag{3-2}$$

All of the other structural equations and restrictions remain as before (Chapter II), except for their initial values.

4. Forms of the New Solutions.

The form of the solution of the m equations (3-2) depends in part on the form of the demand functions, x_{ni}. For the case of constant demand functions, the form of the solution (Appendix A) in matrix notation is:

$$[X] = e^{[B]^{-1}[M]\,(t-t_1)} \left\{ [X(t_1)] - [M]^{-1}[x] \right\} + [M]^{-1}\,[x] \tag{3-3}$$

For the case of variable demand functions which do not remain constant with time, a general form of the solution in matrix notation is

$$[X] = e^{[B]^{-1}\,[M]\,(t-t_1)}\,[X(t_1)] + e^{[B]^{-1}[M]\,t} \int_{t_1}^{t} e^{-[B]^{-1}[M]\,t}\,[-B]^{-1}[x]\,dt. \tag{3-4}$$

At this point we have a model which describes the logistic flows in an enemy logistic system while an attack by the opposing forces is taking place in the area of operations. Chapter IV consists of a discussion of the interdictory considerations based on this model.

CHAPTER IV

INTERDICTION CONSIDERATIONS

IN

THE MODIFIED MODEL

1. General Discussion.

The general topics discussed in this chapter are certain aspects of interdicting an enemy logistic system based on the modified logistic model of Chapter III. The modified model was developed in Chapter III to describe the logistic flows in an enemy logistic system commencing at the time of the assault initiated by the attacking forces in the area of operations, the time of the attack being denoted by t_1. The different trends that any particular flow might take with respect to time after time t_1 depends largely on the demand functions. It is not possible in this paper to show graphically all of the trends that any one variable might take. However, some representative trends are shown graphically in Figure 4(a), (b) and (c), for the primary variables $X_i(t)$, S_i, and a typical $x_{ki}(k \neq i)$, respectively, for the i^{th} depot.

It has been indicated previously (Chapter II) that we can expect to deal with one of four cases, the unlimited case, the depot capacity bound case, the logistic route bound case, or the stockpile limited case. Briefly, let us examine the effect on the modified model when one of the

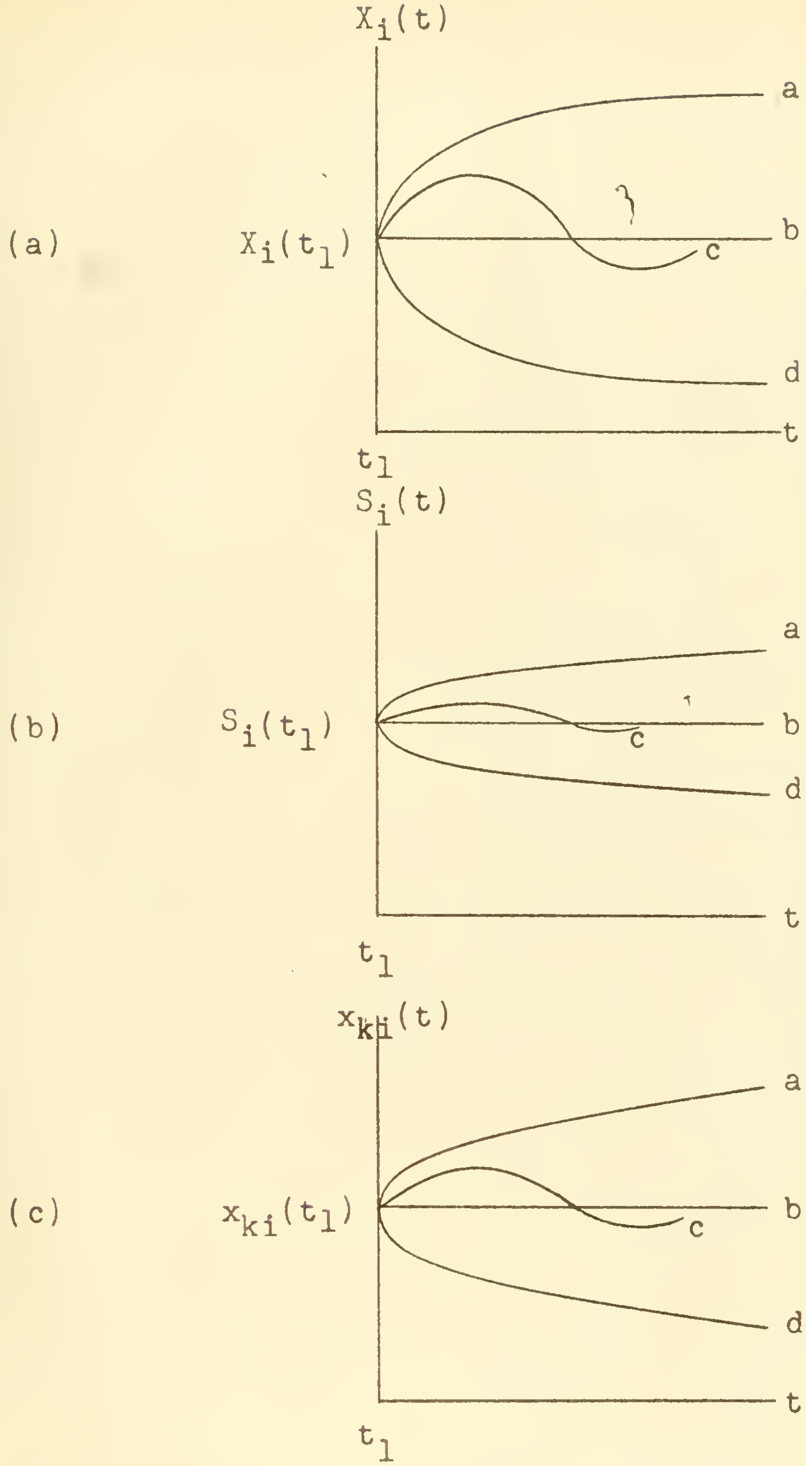

Figure 4. Several Representative Trends of the Variables $X_i(t)$, S_i, and x_{ki}, with Respect to Time.

variables reaches a limiting value. Suppose that $X_i(t)$ for the i^{th} depot reaches its limiting value, $\hat{X}_i$, at some time, say t', later than t_1, and suppose that this is the first variable to reach a limiting value. This depot capacity bound case is shown in Figure 5(a). The input or output flow to the stockpile of that depot, the quantity $\dot{S}_i$, will become zero at time t' and the stockpile size, S_i, will become static. This condition is depicted in Figure 5(b). Furthermore, all logistic route input flows from other depots to this particular depot will become static. This effect is shown in Figure 5(c) for a typical logistic route input flow.

Due to the reciprocity inherent in the model, this sequence of events will affect the other flows in the model, and one or more of these flows will instantly assume new values to compensate for this effect. The model may continue to function normally after these events or the effect may cause one or more of the other variables to reach its limiting value at that time or at a later time. This chain reaction may cause the flows to cascade to their respective limits which may result in ultimate failure of the system to keep pace with the demand functions. It is conceivable that the process may occur at time t_1, but as far as this discussion is concerned, it is assumed that the process occurs later, if it occurs at all. The principles involved

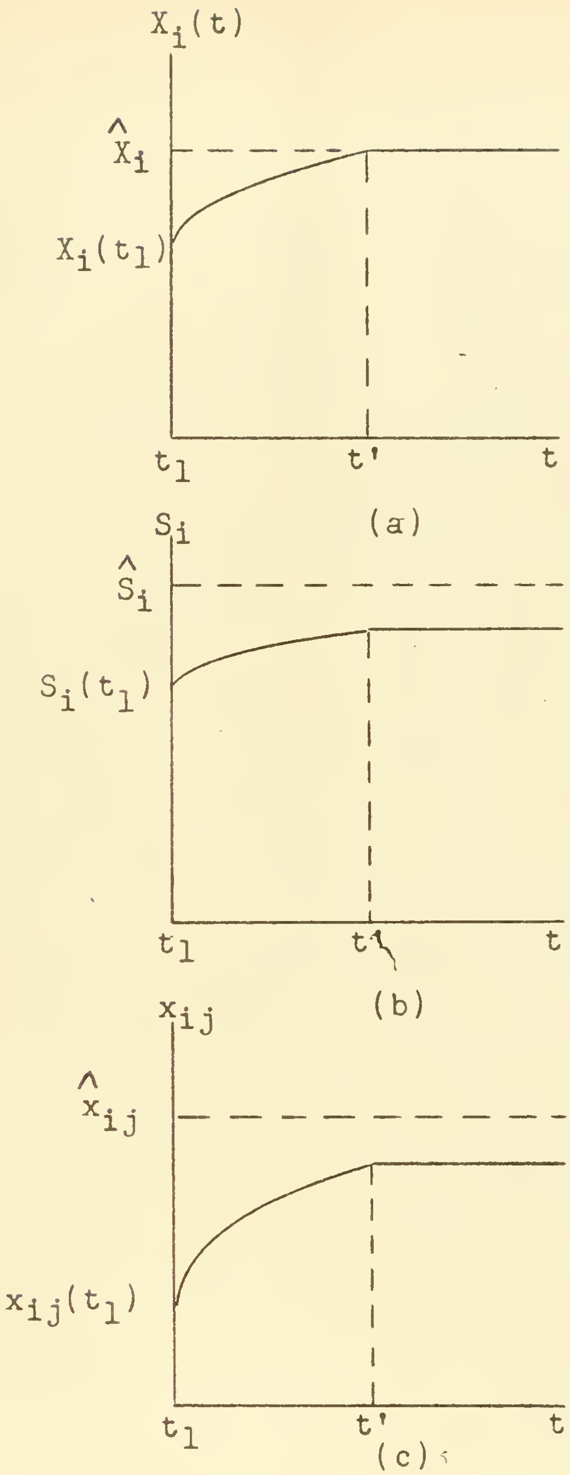

Figure 5. Curves Showing a Typical Depot Capacity Bound Case for the i^{th} Depot

in this limiting process are useful in understanding the interdiction methods to be described later in this chapter.

2. The Interdictory Objective.

In this paper, the primary objective of the attacking commander's interdiction program will be regarded as an attempt to reduce the flows of materiel in the enemy's logistic system by interdiction to such levels that the system will be confined to producing reduced demands. It will be assumed that interdiction will commence at the time of the attack, time t_1. The reduced demands may be represented in the modified model by reduced values of the demand function. The reduced values of the demand functions are denoted by x^*_{ni} $(i = 1, 2, ---, m)$, and are defined by the equations:

$$x^*_{ni} = P_{ni} x_{ni}, \quad i = 1, 2, ---, m \qquad (4\text{-}1)$$
where $\quad 0 \leqq P_{ni} < 1.$

The reduced values of the demand functions are selected by the attacking commander at his discretion. He should select a value for any particular x^*_{ni} such that the enemy forces which would normally rely on the materiel flow x_{ni}, cannot effectively defend themselves with the reduced materiel flow x^*_{ni}. The amount by which the materiel flows x_{ni} are reduced is denoted by R_{ni}. It follows that:

$$R_{ni} = x_{ni} - x^*_{ni} \qquad (4\text{-}2)$$

and

$$R_{ni} = \left[1 - P_{ni}\right] x_{ni}, \quad i = 1, 2, ---, m \qquad (4\text{-}3)$$

3. The Interdiction Methods.

If the flows x_{ni} are to be reduced by R_{ni}, then the other flows in the system must also be reduced in order to preserve the equality of equations (3-2). Let the flows X_i be reduced by the amounts $R\left[X_i(t)\right]$, such that the m equations (3-2) can be expressed in terms of the reduced flows as:

$$\left\{X_i - R\left[X_i(t)\right]\right\} - \sum_{\substack{k=1 \\ k \neq i}}^{m} a_{ki}\left\{X_k - R\left[X_k(t)\right]\right\} - b_i\left\{\dot{X}_i - \dot{R}\left[X_i(t)\right]\right\} = x_{ni} - R_{ni} = x^*_{ni}, \quad (4\text{-}4)$$

or:

$$X_i^* - \sum_{\substack{k=1 \\ k \neq i}}^{m} a_{ki}X_k^* - b_i\dot{X}_i^* = x^*_{ni}, \quad i = 1,2, ---, m \qquad (4\text{-}5)$$

in which:

$$X_i^* = X_i - R\left[X_i(t)\right] \qquad (4\text{-}6)$$

$$x^*_{ki} = a_{ki}X_k^* = a_{ki}\left\{X_k - R\left[X_k(t)\right]\right\} \qquad (4\text{-}7)$$

and

$$\dot{S}_i^* = b_i\dot{X}_i^* = b_i\left\{\dot{X}_i - \dot{R}\left[X_i(t)\right]\right\}. \qquad (4\text{-}8)$$

By subtracting equations (4-4) from equations (3-2), we obtain the m equations

$$R\left[X_i(t)\right] - \sum_{\substack{k=1 \\ k \neq i}}^{m} a_{ki}\, R\left[X_k(t)\right] - b_i\,\dot{R}\left[X_i(t)\right] = R_{ni} \qquad (4\text{-}9)$$

These equations can be solved in terms of the quantities R_{ni} in the same manner as the equations (3-2), and the solution results in the determination of:

$$R\left[X_i(t)\right], \tag{4-10}$$

$$R\left[x_{ki}(t)\right] = a_{ki}\, R\left[X_k(t)\right], \quad k \neq i, \tag{4-11}$$

and

$$\dot{R}\left[S_i(t)\right] = b_i\, \dot{R}\left[X_i(t)\right]. \tag{4-12}$$

From the last equations, (4-12), we may derive:

$$R\left[S_i(t)\right] = R\left[S_i(t_1)\right] + b_i \left\{R\left[X_i(t)\right] - R\left[X_i(t_1)\right]\right\}. \tag{4-13}$$

The attacking commander has the options of interdicting the materiel in the system, the handling facilties of the system, or both. Figures 6, 7 and 8 show graphically typical curves of reduced flows through a depot and in a logistic route, and the amount of reduced materiel in a stockpile, respectively. Consider the reduced flow through the depot (Figure 6). Should the attacking commander elect to interdict the flow of materiel passing through the depot, at time t_1 his interdicting forces should reduce this flow by the amount $R\left[X_i(t_1)\right]$ by destroying materiel as it transits the depot. On the other hand, if he elects to destroy the handling facilities of this depot in order to reduce the flow of materiel through the depot, his interdicting forces should destroy enough of the facilities such that the maximum possible flow of materiel, $\hat{X}_i$, will be reduced to a level $\bar{X}_i(t_1)(= X_i^*(t_1))$. It may be seen in Figure 6 that:

$$\emptyset(t_1) = \hat{X}_i - \bar{X}_i(t_1)$$

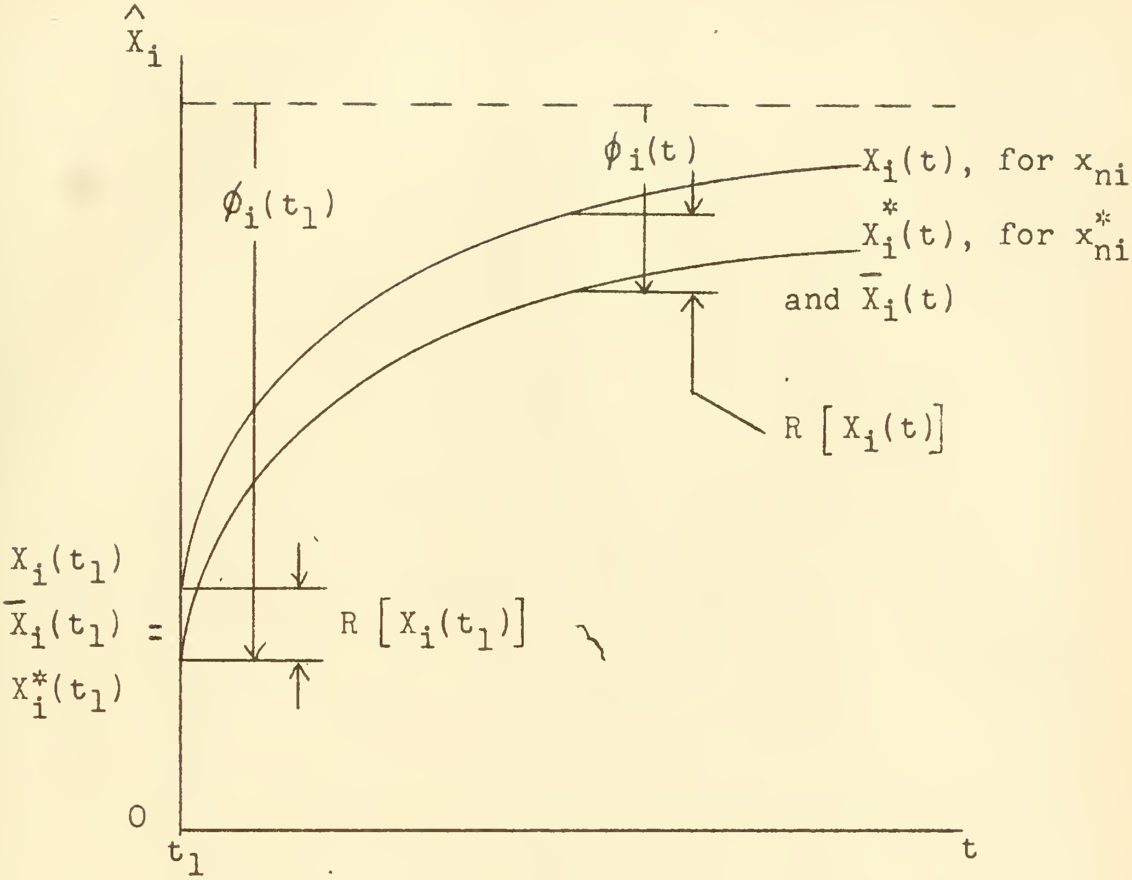

Figure 6. Typical Curve Showing the Reduced Flow of Materiel through the i^{th} Depot (Increasing Flow).

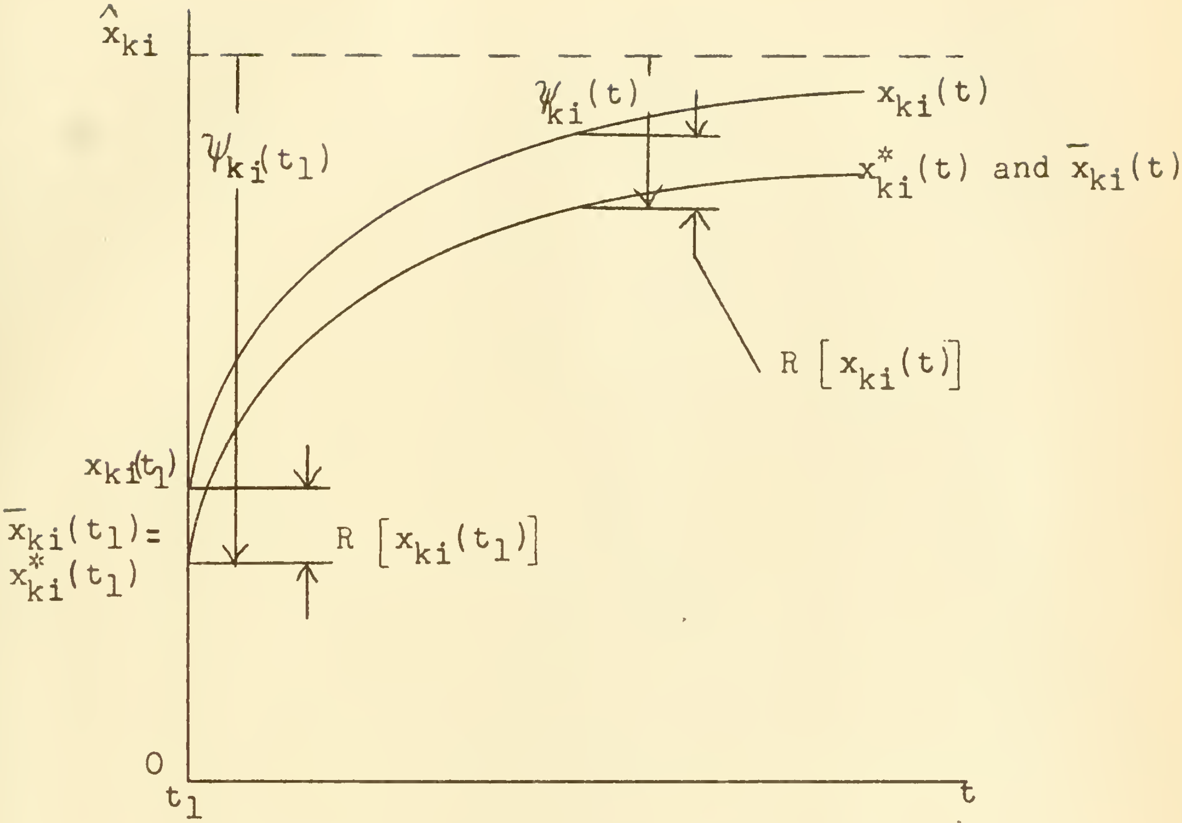

Figure 7. Typical Curves Showing the Reduced Flow of Materiel that is Sent from Depot (i) to Depot (k)(Increasing Flow)

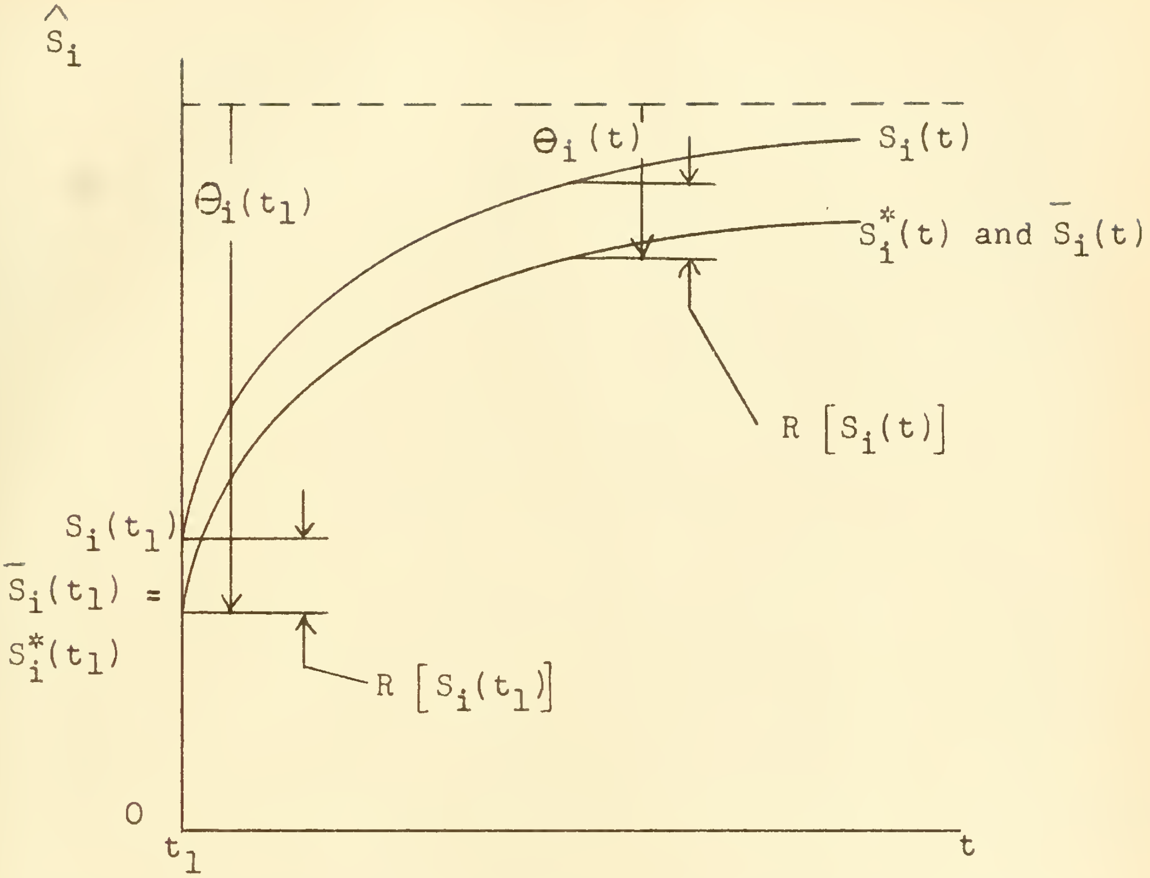

Figure 8. Typical Curves Showing the Reduced Stockpile Size at the i^{th} Depot (Stockpile Accumulating).

As time progresses additional materiel transits the depot and the enemy rebuilds his damaged handling facilities. These facts indicate that additional interdiction is required over time, and these conditions are depicted in Figure 6. Thus it may be deduced that the attacking commander should maintain an interdiction schedule for that depot such that the conditions required by one of the following equations are valid for all t, either:

$$X_i^*(t) = X_i(t) - R\left[X_i(t)\right] , \tag{4-15}$$

or

$$\overline{X}_i(t) \equiv X_i^*(t) = \hat{X}_i - \emptyset_i(t) \tag{4-16}$$

depending on whether he elects to interdict the materiel transiting the depot or the handling facilities of the depot. A similar analysis applies for the flows of materiel over the logistic routes and the materiel stored in the stockpiles, yielding the relationships:

$$x_{ki}^*(t) = x_{ki}(t) - R\left[x_{ki}(t)\right], k \neq i, \tag{4-17}$$

or

$$\overline{x}_{ki}(t) \equiv x_{ki}^*(t) = \hat{x}_{ki} - \psi_{ki}(t), k \neq i, \tag{4-18}$$

and:

$$S_i^*(t) = S_i(t) - R\left[S_i(t)\right] , \tag{4-19}$$

or

$$\overline{S}_i(t) \equiv S_i^*(t) = \hat{S}_i - \theta_i(t) \tag{4-20}$$

This type of analysis would apply to all components of the enemy's logistic system as well.

4. The Magnitude of the Problem.

The interdiction methods analyzed in the previous section are for an ideal situation and there is no assurance that the interdiction objective can be attained. In the first place, the attacking commander may not have sufficient interdictory forces available to be capable of maintaining the magnitude of interdiction effort required. It is possible that the initial effort expended may preclude winning the campaign, which in turn, may impair the success of the strategic war. Furthermore, as soon as the attacking commander interdicts the enemy's logistic system, and regardless of where the system was interdicted, the enemy can readjust certain flows in the system to compensate for the interdiction effects and in accord with any policy that he might exercise. For example, the enemy's policy at any particular depot may be to support his forces in the field by arbitrarily decumulating an accumulating stockpile or it may be to continue building this stockpile at the expense of his forces in the field. There are numerous other options which he may choose. In any case, these effects are produced by changing the constants $a_{ki} (k \neq i)$ and b_i in the model. Since the attacking commander usually has sparse information on these policies, he must re-examine the flows of materiel in the enemy's system between each interdictory effort. These observations indicate a possible application of game theory to the problem.

The author hopes that this approach to the interdiction problem has clarified some of the problems which confront the military commander in planning interdiction problems, and that it will provide him with an implement which he can utilize in planning interdiction campaigns. Chapter V contains a brief discussion of the use of analog computers to calculate certain mathematical cuantities of the models of Chapters II and III.

CHAPTER V

THE USE OF ANALOG COMPUTERS

1. General Discussion.

In solving the systems of equations (2-5) and (3-2) to obtain explicit mathematical solutions, equations (2-9) and (3-4), respectively, it is necessary to perform a matrix inversion, cf. Appendix A. The magnitude of this task is considerable if the number of depots, m, is large, Furthermore, the plotting of the curves, such as those shown in Figure 6, becomes a tedious task for a large number of variables. Much of this work can be avoided by the use of a large-scale analog computer. Such a computer will accept equations (2-5) or (3-2) if one of the equations in a system is expressed in terms of the others. The computer will solve this equation and if a plotter is attached, the plotter will plot the curves for all of the variables desired as a function of time.

The use of analog computing equipment and auxiliary plotters represents a tremendous saving of effort over manual methods of matrix inversion and plotting. It is envisioned in proposing the methods of this paper for planning interdiction campaigns that such equipment would be available to the planner.

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APPENDIX A

SOLUTIONS OF CERTAIN SYSTEMS

OF

DIFFERENTIAL EQUATIONS

1. **A System** of Simple First-Order Form.

In this section we are concerned with the solution of the equations (2-5), namely, the system of m equations of the type:

$$X_i - \sum_{\substack{K=1 \\ K\neq i}}^{m} a_{ki}X_k - b_i\dot{X}_i = 0 \qquad \text{(A-1)}$$

These equations may be stated in matrix form in the following manner:

$$X - AX - B\dot{X} = 0 \qquad \text{(A-2)}$$

where:

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix} \qquad A = \begin{bmatrix} 0 & a_{21} & a_{31} & \cdot\ \cdot & a_{m1} \\ a_{12} & 0 & a_{32} & \cdot\ \cdot & a_{m2} \\ a_{13} & a_{23} & 0 & \cdot\ \cdot & a_{m3} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{1m} & a_{2m} & a_{3m} & \cdot\ \cdot & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 & 0 & \cdot\ \cdot & 0 \\ 0 & b_2 & \cdot\ \cdot & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdot\ \cdot & b_m \end{bmatrix} \quad \text{and} \quad \dot{X} = \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \vdots \\ \dot{X}_m \end{bmatrix}$$

It follows that:

$$X - AX - B\dot{X} = \left[I - A\right] X - B\dot{X} = 0$$

Setting $M = I - A$, we have:

$$MX - B\dot{X} = 0,$$

and $$\dot{X} - B^{-1}MX = 0,$$

or, $$\dot{X} + CX = 0, \text{ where } C = - B^{-1}M. \qquad (A-3)$$

The system of equations (A-3) is similar to the single differential equation:

$$\dot{f} + cf = 0, \qquad (A-4)$$

which has a solution of the form:

$$f(t) = e^{-c(t-t_o)} f(t_o). \qquad (A-5)$$

Equations (A-3) will have the same form of solution as equation (A-4) which results in:

$$X = e^{-c(t-t_o)}X^o, \qquad (A-6)$$

where:

$$C = -B^{-1}M, \text{ and } X^o = \begin{bmatrix} X_1(t_o) \\ X_2(t_o) \\ \vdots \\ X_m(t_o) \end{bmatrix}$$

In order to calculate the matrix $e^{-C(t-t_o)}$ we define:

a. The characteristic matrix of C, namely,

$$f(\lambda) = \lambda I - C.$$

b. The adjoint matrix of $f(\lambda)$, designated by $F(\lambda)$.

c. The characteristic function of the matrix C, namely,

$$\Delta(\lambda) = \left| f(\lambda) \right|.$$

d. λ_r, the characteristic roots of $\Delta(\lambda)$.

If the characteristic roots of the matrix C are all distinct, then $e^{-C(t-t_o)}$ is found by the formula:

$$e^{-C(t-t_o)} = \sum_{r=1}^{m} \frac{e^{-\lambda_r (t-t_o)} F(\lambda_r)}{\Delta^{(1)}(\lambda_r)}, \tag{A-7}$$

where, $\Delta^{(1)}(\lambda_r) \equiv \left\{ \frac{d}{d\lambda} \left[\Delta(\lambda_r) \right] \right\}_{\lambda = \lambda_r}$.

When the characteristic roots of the matrix C are not all distinct, $e^{-C(t-t_o)}$ may be found by the expression:

$$e^{-C(t-t_o)} = \sum_{K=0}^{\infty} \frac{[-C]^K (t-t_o)^K}{K!}. \tag{A-8}$$

2. Solution of the Equations for the Modified Model.

The m equation (3-2) of the type,

$$X_i - \sum_{\substack{K=1 \\ K \neq i}}^{m} a_{ki} X_k - b_i \dot{X}_i = x_{ni}, \tag{A-9}$$

can be expressed in matrix form similarly to equations (A-1), i.e.,

$$\dot{X} + CX = D, \tag{A-10}$$

where: $C = -B^{-1}M = -B^{-1}[I-A]$,

$D = -B^{-1}x$, and

$$x = \begin{bmatrix} x_{n1} \\ x_{n2} \\ \cdot \\ \cdot \\ x_{nm} \end{bmatrix}$$

The solution of equations (A-10) consists of a complementary solution and a particular integral given by:

$$X_c = e^{-C(t-t_1)} X(t_1) \text{ (complementary solution)} \tag{A-11}$$

and

$$X = e^{-Ct}\int_{t_1}^{t} e^{Ct} D\, dt, \text{ (particular integral)} \tag{A-12}$$

The full solution is the sum of the complementary solution and the particular integral, or:

$$X = e^{-C(t-t_1)}X(t_1) + e^{-Ct}\int_{t_1}^{t} e^{Ct} D\, dt. \tag{A-13}$$

If the matrix D consists of constant elements, then the solution (A-13) may be evaluated as:

$$X = e^{-C(t-t_1)}X(t_1) + \left[I - e^{-C(t-t_1)}\right] \left[M^{-1}x\right], \tag{A-14}$$

which is the same as equation (3-3) in the text when the terms are rearranged to give:

$$X = e^{-C(t-t_1)}\left[X(t_1) - M^{-1}x\right] + M^{-1}x,$$

where the matrix $C = - B^{-1}M$. The evaluation of the exponential is the same as before.